

Efficient Monetary Equilibrium: An Overlapping Generations Model with Nonstationary Monetary Policies*

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This paper is concerned with the characterization of public debt policies that are consistent with competitive equilibria in which (i) money is positively priced, and (ii) intertemporal allocation is efficient. The framework used is an overlapping generations model with many goods, agents with two-period lifetimes, and nonstationary tax-transfer policies. We show, under some regularity conditions on such policies that the size of the public debt not growing "too fast" is both necessary and sufficient for the existence of an efficient monetary equilibrium. *Journal of Economic Literature* Classification Numbers: 021, 311, 321. © 1994 Academic Press, Inc.

1. INTRODUCTION

Samuelson [14] introduced the overlapping generations model. The point of this exercise was simple yet profound. If barter is inefficient, an injection of fiat money into the system (initially owned by "generation zero", and subsequently purchased with a physical commodity by each succeeding generation) will restore efficiency. There exists a monetary equilibrium which is efficient.

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Observe that this policy involves no "outside intervention" beyond the initial period. Viewed from another perspective, a public debt has been created, and then there is no attempt to either augment it or to retire it over time. But varying the public debt might have implications for inter-generational distribution. From this perspective (and following here the lead of Balasko and Shell [3]), one might justifiably inquire into such variations. If such distributional changes are desired, *but not at the cost of Pareto optimality*, the following question is fundamental: which sequences of the public debt are compatible with the existence of an efficient monetary equilibrium?¹

We answer this question in a framework which is similar to the one used by Benveniste and Cass [4] in which agents live for two periods, there are many goods, and preferences and endowments are stationary. Even in this simplified structure, solving the problem posed above turns out to be a fairly difficult one. We are looking for a criterion, such that given any tax-transfer policy (in money terms) one should be able to tell from this criterion (with *no other* additional information) whether the policy is consistent with an efficient monetary equilibrium. That is, *the criterion itself must be in terms of the explicit tax-transfer policy sequence announced by the government, the primitive data of the model (preferences and endowments), and nothing else.*

It turns out that the answer is relatively simple to state. To describe it, assume that the barter equilibria of the model are inefficient. This, of course, is the interesting case, since that is precisely the justification for the introduction of public debt into the system in the form of fiat money. If $\{e(t)\}_0^\infty$ describes the tax-transfer policy, and $\{E(t)\}_0^\infty$ the corresponding public debt sequence (satisfying a certain condition on its growth factors), then the tax-transfer policy is consistent with an efficient monetary equilibrium if and only if

$$\sum_{t=0}^T [1/E(t)] \rightarrow \infty \quad \text{as } T \rightarrow \infty \quad (*)$$

That is, given a tax-transfer policy of the kind described above, *the size of the public debt not growing "too fast" is both necessary and sufficient for the existence of an efficient monetary equilibrium.*

¹ We note here that we are demanding more of these tax-transfer policies than just the existence of a monetary equilibrium (efficient or otherwise). This latter issue is, of course, more primitive. If a tax-transfer policy fails the existence criterion, then money must be worthless in equilibrium. Policies which avoid this outcome are called *bonafide* by Balasko and Shell [2, 3]; in a simple overlapping generations model, they have been completely characterized by Mitra [12]. The present question is different: given a tax-transfer policy, how can we tell whether it is consistent with an *efficient* monetary equilibrium?

To illustrate, consider as a special case of our model, the framework of the basic overlapping generations model with *one* perishable good.² A barter equilibrium is then the no-trade equilibrium, and the assumed inefficiency of this equilibrium implies that the marginal rate of substitution of old-age consumption for consumption in youth at the endowment point (call this α) must satisfy $0 < \alpha < 1$.

Note, first, that Samuelson's result with a constant money stock is implied as a consequence of condition (*) being trivially satisfied. There exists an efficient monetary equilibrium in this case.

Perhaps more surprising is the observation that *any* policy of the form $E(t+1) = E(t)\beta$ for $t \geq 0$, where $0 < \beta < 1$, will satisfy our criterion and will, therefore, ensure the existence of an efficient monetary equilibrium.

A tax-transfer policy which creates an additional public-debt of one dollar (say) in each period (that is, $e(t) = 1$ for $t \geq 0$, and so $E(t) = (t+1)$ for $t \geq 0$) will also satisfy the above condition and ensure the existence of an efficient monetary equilibrium.

A debt policy in which the debt grows at a fixed percentage rate in each period (that is, $E(t+1) = E(t)\beta$ for $t \geq 0$, with $\beta > 1$, but $\alpha\beta < 1$ violates (*), and so any monetary equilibrium will be inefficient. Nevertheless, as shown by Mitra [12], a monetary equilibrium exists.

It should be clear that our result is related to the well-known characterizations of "productive" efficiency of Cass [7] and of efficiency (Pareto-optimality) in the overlapping generations models of Bose [5], Balasko and Shell [1], and Okuno and Zilcha [13].³ These characterizations may be employed to deduce the *necessity* of condition (*) in a fairly straightforward way.

By far the harder part of our result is to show that condition (*) is *sufficient* for the existence of an efficient monetary equilibrium (see Theorem 2 for a precise statement). In particular, *the basic difficulty is in proving the existence of a (regular) monetary equilibrium when condition (*) holds*. The main contribution of this paper is in resolving this difficulty. We believe that the existence result (Theorem 2) that is established is new in the literature; the proof might also be of independent interest.

To our knowledge, Millan [11] contains the only definitive statement of existence of efficient monetary equilibria in nonstationary economies. His result assumed a constant money stock but heterogeneous individuals. Our Theorem 2 stated in the context of homogeneous individuals but a varying money stock, complements his.

² For various expositions of this framework, see, for example, Cass and Yaari [6], Shell [15], Gale [10], Cass *et al.* [8], and Wallace [16].

³ Specifically, the proof of our result relies on a version of such a characterization of efficiency, which we state as an independent result (Proposition 1) and relate to the contributions in the above-mentioned papers.

Our analysis is simplified in many respects. One simplification is particularly employed to develop the key results: the assumption of time-separability in preferences. Specifically, it permits us to exploit properties of allocations in a "fictitious two-agent static exchange economy" in which each agent possesses preferences corresponding to each date of the "real" two-period model (see especially Section 4.1 for details).

2. AN OVERLAPPING GENERATIONS ECONOMY

2.1. *The Model*

The framework is a familiar version of an overlapping generations economy.⁴ In each period t , a single agent is born and lives for periods t and $t+1$. At period 0, in addition, there is a single consumer in his "old age." Thus, in each period, there is a single consumer in his "youth" overlapping with a single consumer in his "old age."

There are l intrinsically desirable but perishable goods. The vector $(a, b) \in \mathcal{R}_+^{2l}$ denotes the endowment (of these goods) of any individual over the two periods of his life, and $(c, d) \in \mathcal{R}_+^{2l}$ describes his consumption over those two periods. So at any period t , $d(t)$ is the consumption vector of the old person alive at date t , and $c(t)$ is the consumption vector of the young person overlapping with him. The utility of any person born at any date $t \geq 0$ from a consumption vector (c, d) is $u(c, d)$ where u is a function from \mathcal{R}_+^{2l} to \mathcal{R}_+ . We assume that u is additively separable, so that $u(c, d) = f(c) + g(d)$ for (c, d) in \mathcal{R}_+^{2l} , where f and g are functions from \mathcal{R}_+^l to \mathcal{R}_+ . The old consumer alive at date 0 receives a utility of $g(d(0))$, his consumption at youth being of no relevance to us. A *program* is a sequence $\{c(t), d(t)\}_0^\infty$, satisfying

$$(c(t), d(t)) \in \mathcal{R}_+^{2l} \quad \text{and} \quad c(t) + d(t) = a + b \quad \text{for } t \geq 0.$$

A *tax-transfer policy* is a sequence $\{e(t)\}_0^\infty$ such that

$$e(t) \in \mathcal{R} \quad \text{and} \quad \sum_0^T e(t) \geq 0 \quad \text{for } T \geq 0.$$

Given a tax-transfer policy $\{e(t)\}_0^\infty$, we associate with it a *public debt sequence* $\{E(t)\}_0^\infty$ given by

$$E(t) = \sum_{s=0}^t e(s) \quad \text{for } t \geq 0.$$

⁴ We keep our exposition deliberately terse, and refer the reader to Balasko and Shell [3] for interpretation of the various concepts involved.

A tax-transfer policy $\{e(t)\}_0^\infty$ is called *positive* if its associated public debt sequence $\{E(t)\}_0^\infty$ satisfies $E(t) > 0$ for $t \geq 0$.

Given a tax-transfer policy, $\{e(t)\}_0^\infty$, a *competitive equilibrium* is a sequence $\{c(t), d(t), m(t), n(t), p(t), q(t)\}_0^\infty$ such that

(i) $(c(t), d(t)) \in \mathcal{R}_+^{2l}$, $(m(t), n(t)) \in \mathcal{R}_+^2$, $p(t) \in \mathcal{R}_+^l$, $q(t) \in \mathcal{R}_+$ for $t \geq 0$

(ii) For each $t \geq 0$, $(c(t), d(t+1), m(t), n(t+1))$ is in the set $B(t+1) = \{(c, d, m, n) \in \mathcal{R}_+^{2(l+1)} : p(t)c + q(t)m \leq p(t)a$, and $p(t+1)d + q(t+1)n \leq p(t+1)b + q(t+1)[m + e(t+1)]\}$, and $f(c(t)) + g(d(t+1)) \geq f(c) + g(d)$, among all $(c, d, m, n) \in B(t+1)$. Also, $(d(0), n(0))$ is in $B(0) = \{(d, n) \in \mathcal{R}_+^{l+1} : p(0)d + q(0)n \leq p(0)b + q(0)e(0)\}$, and $g(d(0)) \geq g(d)$ for all $(d, n) \in B(0)$.

(iii) $c(t) + d(t) = a + b$ for $t \geq 0$.

(iv) $m(t) + n(t) = E(t)$ for $t \geq 0$.

A program $\{\hat{c}(t), \hat{d}(t)\}_0^\infty$ is *inefficient* if there is a program $\{c(t), d(t)\}_0^\infty$ satisfying $g(d(0)) \geq g(\hat{d}(0))$, and $f(c(t)) + g(d(t+1)) \geq f(\hat{c}(t) + g(\hat{d}(t+1)))$ for $t \geq 0$, with strict inequality somewhere in the above set of inequalities. A program is *efficient* if it is not inefficient. A program $\{c(t), d(t)\}_0^\infty$ is *stationary* if there is (c, d) such that $(c(t), d(t)) = (c, d)$ for all $t \geq 0$.

A competitive equilibrium is *monetary* if $q_t > 0$ for all $t \geq 0$. A competitive equilibrium is *barter* if $q_t = 0$ for all $t \geq 0$. A *competitive program* is a program generated by some competitive equilibrium. We may also define barter and monetary competitive programs in the obvious way.

A monetary equilibrium is “regular” if it does not wander “too close” to barter equilibrium allocations, nor towards zero consumption in youth. Formally, a monetary equilibrium is *regular* if $\inf_{t \geq 0} c_t \geq 0$ and $\inf_{t \geq 0} \|c_t - \tilde{c}_t\| > 0$ for every $\{\tilde{c}_t\}_0^\infty$ such that $\{\tilde{c}_t, a + b - \tilde{c}_t\}$ is a barter competitive program.⁵

Throughout this paper, we study regular monetary equilibria.

2.2. Assumptions

The following assumptions will be maintained throughout.

(A.1) If $c, c' \in \mathcal{R}_+^l$ and $c' \gg c$, then $f(c') > f(c)$; if $d, d' \in \mathcal{R}_+^l$ and $d' \gg d$, then $g(d') > g(d)$.

(A.2) f, g are continuous and concave functions on \mathcal{R}_+^l .

(A.3) f, g are twice continuously differentiable on \mathcal{R}_{++}^l , with $\nabla f(c) \gg 0$, $\nabla g(d) \gg 0$, and the Hessians $H_f(c)$ and $H_f(d)$ are negative definite for c, d in \mathcal{R}_{++}^l .

⁵ If $x \in \mathcal{R}^n$, $\|x\| = \sum_{i=1}^n |x_i|$.

Denote by A the set $\mathcal{R}'_+ \sim \mathcal{R}'_{++}$, and assume the following regularity condition:

(A.4) (a) If $\bar{c} \in A$, $c^s \in \mathcal{R}'_{++}$ for $s = 1, 2, \dots$, and $c^s \rightarrow \bar{c}$ as $s \rightarrow \infty$, then $\|\nabla f(c^s)\| \rightarrow \infty$. If $\bar{d} \in A$, $d^s \in \mathcal{R}'_{++}$ for $s = 1, 2, \dots$, and $d^s \rightarrow \bar{d}$ as $s \rightarrow \infty$, then $\|\nabla g(d^s)\| \rightarrow \infty$.

(b) If $c \in \mathcal{R}'_+$ and $f(c) > f(0)$, then $c \gg 0$; if $d \in \mathcal{R}'_+$ and $g(d) > g(0)$, then $d \gg 0$.

In what follows, assume without loss of generality that $f(0) = g(0) = 0$.

The above assumptions, apart from postulating standard monotonicity and curvature properties of the utility functions, impose some smoothness and regularity conditions on these functions. Our next assumption is substantive, and sets the backdrop for our study. Recall that we are interested in studying *monetary* equilibrium. In the overlapping generations model, this question is of fundamental significance in a context where equilibria *without* money fail to be efficient. Naturally, this is the case we wish to analyze. Consequently, we assume that

(A.5) Every barter competitive equilibrium is inefficient.

3. THE RESULTS

To prepare for the statements of the results, we will need two preliminary observations, one relating to characterization of efficient programs in this framework, and another to a basic implication of our assumption (A.5) that every barter competitive equilibrium is inefficient.

Our results depend in a crucial way on a complete characterization of efficient programs of the type established by Bose [5] for the case of one perishable good, and Balasko and Shell [1] and Okuno and Zilcha [13] for the general case. (These characterizations of efficiency in overlapping-generations models are closely related to the earlier characterization of "productive" efficiency in the seminal work of Cass [7]). In the context of our model, we state this in Proposition 1 below.

For this purpose, we will need some additional notation. For $(c, d) \in \mathcal{R}'_{++}$, denote $\|\nabla f(c)\|$ by $\lambda(c)$ and $\|\nabla g(d)\|$ by $\mu(d)$. Further, define for $(c, d) \in \mathcal{R}'_{++}$, $F(c) \equiv \nabla f(c)/\lambda(c)$ and $G(d) \equiv \nabla g(d)/\mu(d)$. Note that $\|F(c)\| = \|G(d)\| = 1$.

PROPOSITION 1. *Suppose $\{c(t), d(t)\}_0^\infty$ is a program satisfying $\inf_{t \geq 0} c(t) \gg 0$ and $\inf_{t \geq 0} d(t) \gg 0$. Then $\{c(t), d(t)\}_0^\infty$ is an efficient program if and only if*

$$(i) \quad F(c(t)) = G(d(t)) \quad \text{for } t \geq 0$$

$$(ii) \quad \sum_{t=1}^{\infty} \prod_{s=0}^{t-1} [\lambda(c(s))/\mu(d(s+1))] = \infty.$$

It is useful to relate Proposition 1 to the characterization of efficiency (Pareto Optimality) provided by Balasko and Shell [1]. Given the program $\{c(t), d(t)\}_0^{\infty}$ (with $\inf_{t \geq 0} c(t) \geq 0$ and $\inf_{t \geq 0} d(t) \geq 0$), define

$$\pi(t) = \prod_{s=0}^{t-1} [\lambda(c(s))/\mu(d(s+1))] \quad \text{for } t \geq 1$$

$$P(t) = [\nabla f(c(t))/\pi(t)] \quad \text{for } t \geq 1$$

$$P(0) = P(1)[\lambda(c(1))/\mu(d(1))]$$

If the program $\{c(t), d(t)\}_0^{\infty}$ satisfies our condition (i), then it is "supported" by the price sequence $\{P(t)\}_0^{\infty}$ (unique up to positive scalar multiplication). Thus, $[1/\|P(t)\|] = [\pi(t)/\lambda(c(t))]$ for $t \geq 1$ so that (using (A.3), and noting that $\lambda(c(t))$ is uniformly bounded above) if our condition (ii) is satisfied then

$$\sum_{t=1}^{\infty} (1/\|P(t)\|) = \infty. \quad (\dagger)$$

Thus, the condition of efficiency in Balasko and Shell is satisfied, and $\{c(t), d(t)\}_0^{\infty}$ is efficient.

Conversely, if $\{c(t), d(t)\}_0^{\infty}$ is efficient, then it is clearly efficient in the static economy, so that condition (i) follows. Furthermore, by Balasko and Shell's result, we have condition (\dagger) satisfied, since $\{P(t)\}_0^{\infty}$ price-supports the given efficient program. Thus (using (A.3), and noting that $\lambda(c(t))$ is uniformly bounded below by a strictly positive number), our condition (ii) must also be satisfied.⁶

Next, we note that the set of barter competitive programs is compact in the topology of pointwise convergence, and consequently there exists a stationary barter competitive program $\{c^*(t); d^*(t)\}_0^{\infty}$, with $(c^*(t), d^*(t)) = (c^*, d^*)$ for all $t \geq 0$, such that the utility of each young person, $f(c)$, is minimized over all stationary barter competitive programs with allocation

⁶ To see that Balasko and Shell's result is applicable to our framework, note that it is routine to check that their Property G as well as the crucial Properties C and C' on the Gaussian curvature of indifference curves are satisfied under Assumption (A.3). In fact, (A.3) ensures that the Hessians of f and g are negative definite in the strictly positive orthant, and this enables us to provide a proof, following closely the original technique of Cass [7], without directly using the notion of Gaussian curvature. The reader is referred to Esteban *et al.* [9] for details.

(c, d) . It is easy to check, moreover, that (c^*, d^*) is uniquely defined. Call this the *special barter program*. Note that $(c^*, d^*) \gg 0$.

PROPOSITION 2. *At the allocation (c^*, d^*) corresponding to the special barter program,*

$$\frac{\mu(d^*)}{\lambda(c^*)} > 1.$$

The proof of Proposition 2 will follow immediately from (A.5) and the characterization of efficient programs (Proposition 1).⁷

We may now state our main results.

THEOREM 1. *Let $\{e(t)\}_0^\infty$ be a positive tax-transfer policy. If there exists a regular monetary equilibrium which is efficient, then*

$$(i) \quad \inf_{t \geq 0} \frac{E(t+1)}{E(t)} > 0$$

and

$$(ii) \quad \sum_{t=0}^{\infty} \frac{1}{E(t)} = \infty.$$

Conversely:

THEOREM 2. *Let $\{e(t)\}_0^\infty$ be a positive tax-transfer policy. If*

$$(i) \quad \inf_{t \geq 0} \frac{E(t+1)}{E(t)} > 0$$

$$(ii) \quad \sum_{t=0}^{\infty} \frac{1}{E(t)} = \infty$$

$$(iii) \quad \limsup_{t \rightarrow \infty} \frac{E(t+1)}{E(t)} < \frac{\mu(d^*)}{\lambda(c^*)},$$

then there exists a regular monetary equilibrium which is efficient.

⁷ Observe that $\lambda(c^*)$ and $\mu(d^*)$ can be viewed as the marginal gains of relaxing the budget constraint during youth and old age respectively. The inequality in Proposition 2 means that transfers of wealth from youth to old would improve the utility of each generation, as well as that of generation 0. That is, a Pareto improvement is possible, which is just what (A.5) states. Of course, the opposite inequality would also improve the welfare of each generation in $t \geq 1$. But the initial old person would lose. Therefore, only the inequality in the main text describes the efficiency of the maximal barter program.

Remarks. (1) Taken together, Theorems 1 and 2 do not provide a *complete* characterization, but it is possible to argue that the characterization is tight for a large class of monetary policies. To see this, focus on the condition (ii). In particular, if there exists an efficient equilibrium, condition (ii) states that money supply *cannot* grow exponentially fast. Given that this is so, it is easy to see that a complete description can fail only for “irregular” monetary policies which display exponential growth at a factor exceeding $\mu(d^*)/\lambda(c^*)$ (greater than 1, by Proposition 2), but only “rarely enough” so that condition (ii) is met. To illustrate these remarks more succinctly, consider the following sub-classes of tax-transfer policies:

(A) *Exponential:* $E(t) = E(0)(1 + g)^t$, for $g > -1$. In this case Theorems 1 and 2 together state that a regular efficient monetary equilibrium exists *if and only if* $g \in (-1, 0]$. The case $g = 0$ yields the special case studied by Benveniste and Cass [4].

(B) *Linear:* $E(t) = E(0) + at$, for $a \geq 0$. In this case a regular efficient monetary equilibrium *always* exists.

(2) The heart of the characterization, which we would like the reader to focus attention on, is condition (ii). Although we have not formally stated it here, it is possible to show that *if we assume existence*, a regular monetary equilibrium is efficient *if and only if* (ii) holds. The additional condition (iii) is used to establish existence (more on this below). The *necessity* of this condition was established by Okuno and Zilcha [13]. Our contribution lies in the sufficiency direction, and in proving existence.

4. PROOFS

4.1. A Fictitious Static Economy

It will be useful to consider a certain fictitious static exchange economy, with two consumers (1 and 2) and l goods. The endowment of 1 (resp. 2) is a (resp. b), and the utility function of 1 (resp. 2) is f (resp. g). An *allocation* is a pair $(c, d) \in \mathcal{R}_+^{2l}$ such that $c + d = a + b$. The interpretation is that 1 (resp. 2) consumes c (resp. d).

We start by defining the “contract curve”. Let $K \equiv f(a + b)$. Consider, for each $\theta \in [0, K]$, the problem

$$\left. \begin{array}{l} \max g(d) \\ \text{s.t. } f(c) \geq \theta \\ (c, d) \text{ is an allocation} \end{array} \right\} P(\theta).$$

Next, we present a sequence of preliminary results, without proofs.⁸

LEMMA 1. *For each $\theta \in [0, K]$, there is a unique solution to problem $P(\theta)$; call it $(c(\theta), d(\theta))$. Moreover $(c(\theta), d(\theta))$ is continuous in θ , $(c(\theta), d(\theta)) \geq 0$ for $\theta \in (0, K)$, and $c(\theta) \rightarrow 0$ as $\theta \rightarrow 0$.*

Let $C \equiv \{c(\theta), d(\theta) \mid \theta \in [0, K]\}$. This is the "contract curve," containing the set of all "static Pareto-optimal" allocations.

The following result is standard, and reflects the fact that competitive equilibria of overlapping generations models possess a short-run efficiency property.

LEMMA 2. *Let $\{c(t), d(t)\}_0^\infty$ be a competitive program. Then $(c(t), d(t)) \in C$ for all $t \geq 0$.*

It will be convenient to divide C into two subsets. Note, first, that $(c^*, d^*) \in C$ by Lemma 2. Define $\Omega \equiv \{(c, d) \in C \mid f(c) < f(c^*)\}$, and $\Omega' \equiv \{(c, d) \in C \mid f(c) \geq f(c^*)\}$. Note that for any competitive barter program $\{c(t), d(t)\}_0^\infty$ and any $t \geq 0$, the stationary program $\{\hat{c}(s), \hat{d}(s)\}_0^\infty$ with $(\hat{c}(s), \hat{d}(s)) = (c(t), d(t))$ for all $s \geq 0$ is also a competitive barter program. Using this observation, Lemma 2 and the definition of the special barter program, we may easily deduce

LEMMA 3. *Let $\{c(t), d(t)\}_0^\infty$ be a competitive barter program. Then $(c(t), d(t)) \in \Omega'$ for all $t \geq 0$.*

We continue with a lemma characterizing the "interior" of the contract curve.

LEMMA 4. *Let $(c, d) \geq 0$. Then $(c, d) \in C$ if and only if $F(c) = G(d)$.*

Lemma 4 leads to a simple characterization of allocations along barter competitive programs.

LEMMA 5. *An allocation (c, d) is the outcome of some stationary barter competitive program at some date if and only if $(c, d) \in C$, $(c, d) \geq 0$ and $F(c)(a - c) = 0$.*

Lemmas 3 and 5 lead to the following observation, fundamental to our main argument.

LEMMA 6. *For all $(c, d) \in \Omega$ with $(c, d) \geq 0$, $F(c)(a - c) > 0$.*

⁸ Most of the proofs are fairly straightforward. The reader is referred to Esteban *et al.* [9] for the details.

Next, we pinpoint a *particular* allocation in C . Consider the problem

$$\left. \begin{array}{l} \max f(c) + g(d) \\ \text{s.t. } (c, d) \text{ is an allocation} \end{array} \right\} \text{ (Q).}$$

It can be checked that there is a unique solution (\hat{c}, \hat{d}) to problem (Q), and that $(\hat{c}, \hat{d}) \geq 0$. Such an allocation is often referred to as the *golden rule*.

Following standard methods, it is easy to establish

LEMMA 7. *The golden-rule allocation (\hat{c}, \hat{d}) solves (Q) if and only if $(\hat{c}, \hat{d}) \geq 0$, $F(\hat{c}) = G(\hat{d})$ and $\lambda(\hat{c}) = \mu(\hat{d})$.*

This lemma yields a second important observation:

LEMMA 8. *The golden-rule allocation $(\hat{c}, \hat{d}) \in \Omega$, and in particular $F(\hat{c})(a - \hat{c}) > 0$.*

4.2. Basic Properties of Competitive Equilibria

In this subsection we collect together, without proofs, some basic properties of competitive equilibria.

LEMMA 9. *Suppose $\{e(t)\}_0^\infty$ is a tax-transfer policy and $\{c(t), d(t), m(t), n(t), p(t), q(t)\}_0^\infty$ is a competitive equilibrium; then*

- (i) (a) $p(t)c(t) + q(t)m(t) = p(t)a$, and $p(t+1)d(t+1) + q(t+1)n(t+1) = p(t+1)b + q(t+1)[m(t) + e(t+1)]$ for $t \geq 0$.
- (b) $p(0)d(0) + q(0)n(0) = p(0)b + q(0)e(0)$.
- (ii) $q(t)n(t) = 0$ for $t \geq 0$.
- (iii) $p(t) > 0$ for $t \geq 0$.
- (iv) *If $q(t) = 0$ for some $t \geq 0$, then $q(t+1) = 0$.*

In words, the result states that (i) Budget constraints hold with equality; (ii) If money is valued, it is *not* demanded in old age; (iii) For each period, there is some intrinsically desirable commodity which is positively priced; (iv) If money is worthless today, it is worthless tomorrow.

Next, we specialize to the case of *positive* tax-transfer policies. In this case money is either *always* worthless or *never* worthless.

LEMMA 10. *Suppose $\{e(t)\}_0^\infty$ is a positive tax-transfer policy, and $\{c(t), d(t), m(t), n(t), q(t)\}_0^\infty$ is a competitive equilibrium. Then either $q(t) = 0$ for all $t \geq 0$, or $q(t) > 0$ for all $t \geq 0$.*

LEMMA 11. Suppose $\{e(t)\}_0^\infty$ is a positive tax-transfer policy, and $\{c(t), d(t), m(t), n(t), p(t), q(t)\}_0^\infty$ is a monetary competitive equilibrium; then

- (i) $q(t) E(t) = p(t)[a - c(t)]$ for $t \geq 0$
- (ii) $(c(t), d(t)) \geq 0$ for $t \geq 0$
- (iii) $F(c(t)) = G(d(t)) = p(t)/\|p(t)\|$ for $t \geq 0$.

If a competitive equilibrium $\{c(t), d(t), m(t), n(t), p(t), q(t)\}_0^\infty$ is monetary, then $m(t) = E(t)$ and $n(t) = 0$. Thus we can describe the equilibrium by $\{c(t), d(t), p(t), q(t)\}_0^\infty$. It is also clear from the definition of a competitive equilibrium that if we define

$$p'(t) = p(t)/\|p(t)\|, \quad q'(t) = q(t)/\|p(t)\| \quad \text{for } t \geq 0$$

then $\{c(t), d(t), p'(t), q'(t)\}_0^\infty$ is also a monetary competitive equilibrium. Further, we have, for $t \geq 0$, $p'(t) = F(c(t)) = G(d(t))$ and

$$\frac{q'(t+1)}{q'(t)} = \frac{\lambda(c(t))}{\mu(d(t+1))}.$$

In the rest of the paper, whenever we describe a monetary equilibrium, we will use the prices $p'(t), q'(t)$ as given above. However, to ease the writing, we will drop the primes henceforth.

Our final preliminary result is

LEMMA 12. Suppose that $\{e(t)\}_0^\infty$ is a positive tax-transfer policy and $\{c(t), d(t), p(t), q(t)\}_0^\infty$ is a monetary competitive equilibrium. Then $F(c(t))(a - c(t)) > 0$.

4.3. Proof of Theorem 1

Proof. Using Lemma 11, we have

$$\frac{q(t+1) E(t+1)}{q(t) E(t)} = \frac{p(t+1)(a - c(t+1))}{p(t)(a - c(t))}.$$

Now, recalling our convention as to the prices, we can write

$$\frac{E(t+1)}{E(t)} = \frac{\mu(d(t+1)) F(c(t+1))(a - c(t+1))}{\lambda(c(t)) F(c(t))(a - c(t))}.$$

Note that since $\inf_{t \geq 0} c(t) \geq 0$, we can find $0 < M < \infty$ such that $\lambda(c(t)) F(c(t))(a - c(t)) = \nabla f(c(t))(a - c(t)) \leq \nabla f(c(t))a \leq M$ for $t \geq 0$. We now claim that there exists $M' > 0$ such that $\mu(d(t+1)) F(c(t+1))(a - c(t+1)) \geq M'$ for $t \geq 0$.

For $t \geq 0$, we have $F(c(t))(a - c(t)) > 0$ by Lemma 12, and so $G(d(t))(d(t) - b) > 0$. This means that $\nabla g(d(t))b < \nabla g(d(t))d(t) \leq g(d(t)) \leq g(a + b)$. Then, by Assumption (A.4), we must have $\inf_{t \geq 0} d(t) \geq 0$. Thus, $\mu(d(t))$ is bounded (above and below). Consequently, if our claim is not true, we must have $F(c(t+1))(a - c(t+1)) \rightarrow 0$ for some subsequence of t . Note that $(c(t+1), d(t+1)) \in C$ for all t , so that the convergence of $F(c(t+1))(a - c(t+1)) \rightarrow 0$ along a subsequence, coupled with the fact that $\inf_{t \geq 0} (c(t+1), d(t+1)) \geq 0$, means that there exists a further subsequence for which $(c(t+1), d(t+1)) \rightarrow (c, d) \in C$, where $(c, d) \geq 0$ and $F(c)(a - c) = 0$. But then, by Lemma 5, (c, d) is the allocation along some stationary barter competitive program. This contradicts our supposition that $\{c(t), d(t)\}_0^\infty$ is regular, and establishes our claim, and hence (i).

To prove (ii), use Lemma 11 (and our convention as to the prices) to write, for $t \geq 0$,

$$q(t) E(t) = F(c(t))(a - c(t)).$$

Now, $F(c(t))(a - c(t)) \leq F(c(t))a \leq \|a\| \equiv A < \infty$. Thus, we have

$$\frac{1}{E(t+1)} \geq \frac{q(t+1)}{A} = \frac{q(0)}{A} \prod_{s=0}^t \frac{\lambda(c(s))}{\mu(d(s+1))}.$$

Noting that $\inf_{t \geq 0} (c(t), d(t)) \geq 0$ and applying Proposition 1(ii), (ii) follows immediately.

4.4. Proof of Theorem 2

First, we state a couple of additional lemmas.

LEMMA 13. *Suppose that $\{e(t)\}_0^\infty$ is a positive tax-transfer policy. Suppose that for some $t \geq 0$, we have $(c(t+1), d(t+1)) \in C$ with $(c(t+1), d(t+1)) \geq 0$ and $F(c(t+1))(a - c(t+1)) > 0$. Then there is $(c(t), d(t)) \in \Omega$ with $F(c(t))(a - c(t)) \geq 0$, satisfying*

$$\frac{E(t+1)}{E(t)} = \frac{\mu(d(t+1)) F(c(t+1))(a - c(t+1))}{\lambda(c(t)) F(c(t))(a - c(t))}. \tag{1}$$

Proof. Note that $[E(t+1)/E(t)] > 0$ and $B(t) \equiv \mu(d(t+1)) F(c(t+1))(a - c(t+1)) > 0$ by assumption. So $B(t)[E(t)/E(t+1)] > 0$. Consider the solution $(c(\theta), d(\theta))$ to problem $P(\theta)$ for $\theta \in (0, f(c^*))$. As $\theta \rightarrow f(c^*)$, $c(\theta) \rightarrow c^*$, and so $\nabla f(c(\theta))(a - c(\theta)) \rightarrow \nabla f(c^*)(a - c^*) = 0$, by Lemma 5. As $\theta \rightarrow 0$, $c(\theta) \rightarrow 0$, and so $\nabla f(c(\theta))(a - c(\theta)) \rightarrow \infty$. So there is $\bar{\theta} \in (0, f(c^*))$ such that $\nabla f(c(\bar{\theta}))(a - c(\bar{\theta})) = B(t)[E(t)/E(t+1)]$. Defining $(c(t), d(t)) \equiv (c(\bar{\theta}), d(\bar{\theta}))$, we see that (1) is satisfied and that $(c(t), d(t)) \in \Omega$.

LEMMA 14. Suppose $\{e(t)\}_0^\infty$ is a positive tax-transfer policy. Suppose $\{c(t), d(t)\}_0^\infty$ is a program which satisfies $(c(t), d(t)) \geq 0$, and $(c(t), d(t)) \in \Omega$ for $t \geq 0$. If there is a sequence of positive scalars $\{q(t)\}_0^\infty$ such that

$$\text{and } \left. \begin{aligned} q(0) > 0, \quad q(t) &= q(0) \prod_{s=0}^{t-1} [\lambda(c(s))/\mu(d(s+1))] \quad \text{for } t \geq 1 \\ q(t) E(t) &= F(c(t))(a - c(t)) \quad \text{for } t \geq 0 \end{aligned} \right\} \quad (2)$$

then $\{c(t), d(t), F(c(t)), q(t)\}_0^\infty$ is a monetary equilibrium (with $(m(t), n(t)) = (E(t), 0)$ for $t \geq 0$).

The proof of Lemma 14 is straightforward and is, therefore, omitted. We now prove Theorem 2.

Proof (of Theorem 2). Fix any $T \geq 1$. Define $(c^T(T), d^T(T)) \equiv (\hat{c}, \hat{d})$, the golden-rule allocation. By Lemma 8, $F(\hat{c})(a - \hat{c}) > 0$. So we may use Lemma 13 repeatedly to define $\{c^T(t), d^T(t)\}_0^T$ in Ω such that

$$\frac{E(t+1)}{E(t)} = \frac{\mu(d^T(t+1)) F(c^T(t+1))(a - c^T(t+1))}{\lambda(c^T(t)) F(c^T(t))(a - c(t))}. \quad (3)$$

Define a sequence $\{q^T(t)\}_0^T$ by

$$\left. \begin{aligned} q^T(0) &\equiv [F(c^T(0))(a - c^T(0))]/E(0), \quad \text{and for } t = 1, \dots, T \\ q^T(t) &\equiv q^T(0) \prod_{s=0}^{t-1} [\lambda(c^T(s))/\mu(d^T(s+1))] \end{aligned} \right\} \quad (4)$$

Note that $q^T(t) > 0$ for $t = 0, \dots, T$. Combining (3) and (4),

$$q^T(t) E(t) = F(c^T(t))(a - c^T(t)) \quad (5)$$

for $t = 0, \dots, T$. Now, because $(c^T(t), d^T(t)) \in \Omega$, $g(d^T(t)) \geq g(d^*)$, and so there is $\bar{d} \geq 0$ with $d^T(t) \geq \bar{d}$ for all T, t . So there is $0 < B < \infty$ such that $\mu(d^T(t)) \leq B$ for all T, t . Using this information, $\mu(d^T(t+1)) F(c^T(t+1))(a - c^T(t+1)) \leq B \|a\|$ for all T, t . Using condition (i) of the theorem, there is $\bar{B} > 0$ such that $[E(t+1)/E(t)] \geq \bar{B}$ for all t . Combining all this information with (3), we get for all T, t ,

$$\nabla f(c^T(t))(a - c^T(t)) \leq [B \|a\|/\bar{B}].$$

Thus $\nabla f(c^T(t))a \leq \nabla f(c^T(t))c^T(t) + [B \|a\|/\bar{B}] \leq f(c^T(t)) + [B \|a\|/\bar{B}] \leq f(a+b) + [B \|a\|/\bar{B}]$. Consequently, we can find $c^* \geq \bar{c} \geq 0$, such that $c^T(t) \geq \bar{c}$ for all T, t .

Using the fact that $(c^T(t), d^T(t)) \geq (\tilde{c}, \tilde{d}) \geq 0$ for all T, t , we can use (4) to obtain a sequence $\{A(t)\}_0^\infty$, with $A(t) > 0$, such that

$$0 < q_t^T \leq A(t) \quad \text{for all } T \geq t. \quad (6)$$

Combining (6) with the fact that $(c^T(t), d^T(t))$ are uniformly bounded, we can use a Cantor diagonal argument to extract a subsequence of T (retain notation), such that for each t ,

$$(c^T(t), d^T(t), q^T(t)) \rightarrow (\hat{c}(t), \hat{d}(t), \hat{q}(t)) \quad \text{as } T \rightarrow \infty. \quad (7)$$

Using (4), we then get

$$\hat{q}(0) = [F(\hat{c}(0))(a - \hat{c}(0))]/E(0) \quad (8)$$

and for $t \geq 1$,

$$\hat{q}(t) = \hat{q}(0) \prod_{s=0}^{t-1} [\lambda(\hat{c}(s))/\mu(\hat{d}(s+1))]$$

Using (5), we also get

$$\hat{q}(t) E(t) = F(\hat{c}(t))(a - \hat{c}(t)) \quad \text{for } t \geq 0. \quad (9)$$

Since $(c^T(t), d^T(t)) \geq (\tilde{c}, \tilde{d})$ for all T, t ,

$$(\hat{c}(t), \hat{d}(t)) \geq (\tilde{c}, \tilde{d}) \quad \text{for } t \geq 0. \quad (10)$$

We now show that $\{\hat{c}(t), \hat{d}(t), F(\hat{c}(t)), \hat{q}(t)\}_0^\infty$ is a regular efficient monetary equilibrium. To this end, define

$$K \equiv \sup_{t \geq 0} \frac{E(t+1)}{E(t)}; \quad V \equiv \limsup_{t \rightarrow \infty} \frac{E(t+1)}{E(t)}. \quad (11)$$

Using condition (ii) of the theorem, choose $\tilde{V} > \max\{V, 1\}$ such that $\tilde{V} < [\mu(d^*)/\lambda(c^*)]$. Because $\tilde{V} > V$, we can find N such that

$$\frac{E(t+1)}{E(t)} < \tilde{V} \quad \text{for } t \geq N. \quad (12)$$

Pick $\tilde{\theta} \in (0, f(c^*))$ such that for all $\theta \in [\tilde{\theta}, f(c^*)]$,

$$\mu(d(\theta))/\lambda(c(\theta)) \geq \tilde{V}. \quad (13)$$

(Use Lemma 1 and the continuity of $\mu(\cdot)$ and $\lambda(\cdot)$ on \mathcal{R}'_{++} to do this.)

Note that because $\lambda(\hat{c}) = \mu(\hat{d})$, we have $f(\hat{c}) < \tilde{\theta}$. Also, observe that by Lemma 6, there exists $M > 0$ such that

$$\mu(d(\theta)) F(c(\theta))(a - c(\theta)) \geq M \quad \text{for } \theta \in (0, \tilde{\theta}]. \quad (14)$$

Now, we claim that for all $t \geq N$,

$$\lambda(\hat{c}(t)) F(\hat{c}(t))(a - \hat{c}(t)) \geq \frac{M}{2K}. \quad (15)$$

For, if (15) is violated for some $t \geq N$, then we can find $T > s$ such that

$$\lambda(c^T(t)) F(c^T(t))(a - c^T(t)) < \frac{M}{K}. \quad (16)$$

Now either $f(c^T(t+1)) \leq \tilde{\theta}$ or $f(c^T(t+1)) > \tilde{\theta}$. If the former, then by (14),

$$\mu(d^T(t+1)) F(c^T(t+1))(a - c^T(t+1)) \geq M. \quad (17)$$

But then, combining (3), (16), and (17), $[E(t+1)/E(t)] > K$, a contradiction. So it must be the case that $f(c^T(t+1)) > \tilde{\theta}$. So, by (13),

$$\frac{\mu(d^T(t+1))}{\lambda(c^T(t+1))} \geq \tilde{v}. \quad (18)$$

Using (12), (16), and (18) in (3), we get

$$\lambda(c^T(t+1)) F(c^T(t+1))(a - c^T(t+1)) < \frac{M}{K} \tilde{v} \frac{\lambda(c^T(t+1))}{\mu(d^T(t+1))} \leq \frac{M}{K}. \quad (19)$$

Note that (19) is the same inequality for $t+1$ as (16) was for t . This process can therefore be repeated to get, ultimately, $f(c^T(T)) > \tilde{\theta}$. But $c^T(T) = \hat{c}$, and we have already noted that $f(\hat{c}) < \tilde{\theta}$. This contradiction establishes our claim (15).

Using (15), we certainly have

$$F(\hat{c}(t))(a - \hat{c}(t)) > 0 \quad \text{for } t \geq N. \quad (20)$$

It follows from (20) and (9) that $\hat{q}(t) > 0$ for $t \geq N$, so that $\hat{q}(0) > 0$ and indeed $\hat{q}(t) > 0$ for all $t \geq 0$, by (8). Thus we have $F(\hat{c}(t))(a - \hat{c}(t)) > 0$ for all $t \geq 0$, and so by (15),

$$\inf_{t \geq 0} \lambda(\hat{c}(t)) F(\hat{c}(t))(a - \hat{c}(t)) > 0. \quad (21)$$

Because $\lambda(c) F(c)(a - c)$ is continuous on \mathcal{R}'_+ , it follows from (21) that

$$\inf_{t \geq 0} \|\hat{c}(t) - c^*\| > 0. \quad (22)$$

So, in particular, using the construction of $\{\hat{c}(t), \hat{d}(t)\}_0^\infty$ and (22), we have $(\hat{c}(t), \hat{d}(t)) \in \Omega$ for all $t \geq 0$. We may further conclude from (22) that

$$\inf_{t \geq 0} \|\hat{c}(t) - c\| > 0 \quad (23)$$

for all c such that $(c, a + b - c)$ is the allocation along some barter competitive program at some date. Combining (22) and (23),

$$\inf_{t \geq 0} \|\hat{c}(t) - c(t)\| > 0$$

for all $\{c(t)\}_0^\infty$ such that $\{c(t), a + b - c(t)\}_0^\infty$ is a barter competitive program. Therefore $\{\hat{c}(t), \hat{d}(t)\}_0^\infty$ is regular.

Using (8), (9), (10), the fact that $(\hat{c}(t), \hat{d}(t)) \in \Omega$ for all t , and $\hat{q}(t) > 0$ for all $t \geq 0$, we know that $\{\hat{c}(t), \hat{d}(t), F(\hat{c}(t)), \hat{q}(t)\}_0^\infty$ is a monetary equilibrium, by Lemma 14.

Finally, we verify that $\{\hat{c}(t), \hat{d}(t)\}_0^\infty$ is efficient. Using (8) and (9),

$$\prod_{s=0}^{t-1} [\lambda(\hat{c}(s))/\mu(\hat{d}(s+1))] = \frac{\hat{q}(t)}{\hat{q}(0)} = \frac{F(\hat{c}(t))(a - \hat{c}(t))}{\hat{q}(0)E(t)}. \quad (24)$$

Using (10) and (21), there is $\eta > 0$ such that

$$F(\hat{c}(t))(a - \hat{c}(t)) \geq \eta \quad \text{for } t \geq 0. \quad (25)$$

Using (24) and (25), we get

$$\prod_{s=0}^{t-1} [\lambda(\hat{c}(s))/\mu(\hat{d}(s+1))] \geq \frac{\eta}{\hat{q}(0)E(t)}. \quad (26)$$

Now, using condition (iii) of the Theorem along with (26), and noting (10), $\{c(t), d(t)\}_0^\infty$ is efficient by Proposition 1.

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